Comments on a paper by E. Jakeman and E. R. Pike, `The intensity-fluctuation distribution of Gaussian light'

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LETTERS TO THE EDITOR

Comments on a paper by E. Jakeman and E. R. Pike, 'The intensity-fluctuation distribution of Gaussian light'†

Abstract. In response to a recent paper by Jakeman and Pike, it is pointed out that the classical and quantum-mechanical treatments of the fluctuations of the integrated light intensity are equivalent. The choice of the observable is dictated not by theoretical conditions, but by reference to what is usually measured.

In a recent paper Jakeman and Pike (1968) have discussed the fluctuations of the time-integrated intensity of Gaussian light. They define two random processes

$$E(T) = \int_0^T \mathscr{E}^{(+)}(r, t)\mathscr{E}^{(-)}(r, t)\, \mathrm{d}t$$

and

$$E_c(T) = \int_0^T \mathscr{E}^2(r, t)\, \mathrm{d}t$$

where $\mathscr{E}(r, t)$ is the electric field (treated as a scalar) and $\mathscr{E}^{(+)}(r, t)$ and $\mathscr{E}^{(-)}(r, t)$ are its positive and negative frequency components. Although both the variables $\mathscr{E}^{(+)}(r, t)\mathscr{E}^{(-)}(r, t)$ and $\mathscr{E}^2(r, t)$, and therefore $E(T)$ and $E_c(T)$, are c numbers in Jakeman and Pike's treatment, which is classical throughout, the authors characterize $\mathscr{E}^2(r, t)$ as a 'classical form' and $\mathscr{E}^{(+)}(r, t)\mathscr{E}^{(-)}(r, t)$ as quantum-mechanical. For example, they write that the probability density of $E_c(T)$ "would be the intensity-fluctuation distribution of Gaussian light if the classical form $\mathscr{E}^2(r, t)$ were taken for the intensity instead of the quantum-mechanical formula". They also make the statement: "The classical formula has been used incorrectly in the past; we note, for instance, that equation (6.13) of the review paper by Mandel and Wolf (1965) is based on the classical instead of the quantum-mechanical distribution." The implication of this statement is that the derivation of the quoted equation in our review article is based on the wrong formula, namely the distribution of the variate $E_c(T)$, and that the analysis in our paper constitutes a 'classical' treatment of the problem, which is different from the authors' 'quantum-mechanical' one. We should like to point out that both these conclusions are invalid, and comment briefly on the questions raised.

While $\mathscr{E}^{(+)}(r, t)\mathscr{E}^{(-)}(r, t)$ is, as a rule, a more meaningful measure than $\mathscr{E}^2(r, t)$ of the instantaneous intensity of a light beam, the choice of the former over the latter has nothing at all to do with the difference between quantum-mechanical and classical approaches to the problem, but with considerations of what is usually measured in practice. The choice of $\mathscr{E}^{(+)}(r, t)\mathscr{E}^{(-)}(r, t)$ is already dictated by purely classical considerations, and was in fact made in the early classical treatment of optical coherence problems (Wolf 1954, 1955, Blanc-Lapierre and Dumontet 1955). The quantum theory of coherence, developed a decade later (Glauber 1963 a), confirms this choice. All practical measurements involve averages over time intervals large compared with typical periods of the light, so that only the envelope of the real optical field $\mathscr{E}(r, t)$ is measured. It is partly because the modulus of the analytic signal $\mathscr{E}^{(+)}(r, t)$ (Gabor 1946) is a measure of the wave envelope that the complex $\mathscr{E}^{(+)}(r, t)$ representation was first introduced into optical coherence theory (in this connection see also Mandel 1967). However, it is possible to obtain the same results by working with the square of the real field $\mathscr{E}^2(r, t)$, and averaging over several periods of the light (cf. Purcell 1956, Mandel 1959).

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Secondly, it is a common misunderstanding that the results of a classical and a quantum-mechanical treatment of the electromagnetic field are necessarily different, and that the classical treatment is therefore wrong. This problem has been widely discussed in the literature (Sudarshan 1963, Glauber 1963 b, Klauder 1966, Klauder and Sudarshan 1968), and it has been shown that, whenever the density matrix of the field in the diagonal coherent-state representation is a positive functional, the results of all quantum-mechanical free-field calculations involving normally ordered operators are identical with the results of a corresponding classical calculation. This conclusion applies even to such seemingly unclassical situations as the photoelectric counting of very weak light beams (Mandel and Wolf 1966).

By way of illustration we point out that, for the problem treated by Jakeman and Pike, the density operator $\hat{\rho}$ of the field can be expressed in the form (cf. Mandel and Wolf 1965)

$$\hat{\rho} = \int \prod_{\lambda} \frac{1}{\pi \langle n_\lambda \rangle} \exp\left(- \frac{|v_\lambda|^2}{\langle n_\lambda \rangle} \right) |v_\lambda\rangle\langle v_\lambda| \, \mathrm{d}^2 v_\lambda \tag{1}$$

where $|v_\lambda\rangle$ is the coherent state labelled by the complex number v_λ. (We designate all operators by the caret.) The characteristic function for the normally ordered moments of the operator

$$\hat{E}(T) = \int_0^T \hat{\mathscr{E}}^{(+)}(r, t)\hat{\mathscr{E}}^{(-)}(r, t) \, \mathrm{d}t \tag{2}$$

is then given by

$$C(x) = \mathrm{Tr}(:\exp(\mathrm{i}x\hat{E}):\hat{\rho}) \tag{3}$$

where : : denotes the normally ordered form of the operator within the colons. With the aid of equations (1) and (2), and by making explicit use of the fact that $|v\rangle$ and $\langle v|$ are right and left eigenstates of $\hat{\mathscr{E}}^{(+)}(r, t)$ and $\hat{\mathscr{E}}^{(-)}(r, t)$, respectively, we find that

$$C(x) = \int \left[\exp\left\{ \mathrm{i}x \int_0^T \mathscr{E}^{(+)}(r, t)\mathscr{E}^{(-)}(r, t) \, \mathrm{d}t \right\} \right] \prod_{\lambda} \frac{1}{\pi \langle n_\lambda \rangle} \exp\left(- \frac{|v_\lambda|^2}{\langle n_\lambda \rangle} \right) \mathrm{d}^2 v_\lambda \tag{4}$$

where $\mathscr{E}^{(+)}(r, t)$ and $\mathscr{E}^{(-)}(r, t)$ are the right and left eigenvalues of $\hat{\mathscr{E}}^{(+)}(r, t)$ and $\hat{\mathscr{E}}^{(-)}(r, t)$, respectively. By introducing the function

$$P(E') = \int \delta(E' - E) \prod_{\lambda} \frac{1}{\pi \langle n_\lambda \rangle} \exp\left(- \frac{|v_\lambda|^2}{\langle n_\lambda \rangle} \right) \mathrm{d}^2 v_\lambda \tag{5}$$

we can express equation (4) in the more compact form

$$C(x) = \int_0^\infty P(E) \exp(\mathrm{i}xE) \, \mathrm{d}E. \tag{6}$$

But this is just the characteristic function of E treated as a classical random process, with probability density given by equation (5). These considerations show explicitly that there is no difference between the results obtained by classical and by quantum-mechanical calculations in this case. It is, of course, for this reason that Jakeman and Pike were able to perform a classical calculation (while announcing a quantum-mechanical treatment in the abstract), and arrive at the correct result.

Finally, we should like to point out that the different variates v_λ appearing in the expansion of $\mathscr{E}^{(+)}(r, t)$ (the α_k of Jakeman and Pike) are not "uncorrelated by definition" for thermal (or Gaussian) light, as the authors state. On the contrary, independence of the mode amplitudes usually implies some very special properties for the field in question. Since Jakeman and Pike do not specify the 'normal modes' in terms of which they expand the field $\mathscr{E}^{(+)}(r, t)$ (their equations (2) and (3)), the resulting form of the field cannot be determined. However, if we suppose that the 'normal modes' in question are the usual plane-wave eigenmodes of a large cube, then the field described by the authors' equation (4)

is homogeneous, i.e. it has no preferred origin of space. It does not therefore represent light emanating from a scattering centre, as Jakeman and Pike imply in their introduction.

Department of Physics and Astronomy,
University of Rochester,
Rochester,
New York,
U.S.A.

L. MANDEL
E. WOLF
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The objections raised by Mandel and Wolf to the use of the words 'quantum-mechanical' in our paper (Jakeman and Pike 1968, to be referred to as JP) follow a familiar pattern (see for example their criticism (Mandel and Wolf 1966) of Morawitz (1965)) and have been much discussed in the literature. Their arguments can be broadly divided into two classes: those relating to the choice of the measured quantity $E(T)$ and those concerned with the description of the field and the ensuing distribution-function calculation.

It is not possible to prove classically that the probability of emission of a photoelectron in time Δt, which is the quantity usually measured in photon-counting experiments, is proportional to the c number $\mathscr{E}^+(t)\mathscr{E}^-(t)$ (Einstein 1905). Arguments involving the classical envelope function (analytic signal) cannot be rigorous since the quantum theory of photodetection by annihilation (Glauber 1963) shows that the associated envelope operator $\{\hat{\mathscr{E}}^+(t)\hat{\mathscr{E}}^-(t)+\hat{\mathscr{E}}^-(t)\hat{\mathscr{E}}^+(t)\}/2$, whose use has been suggested for noise-current spectra at optical frequencies by Eckstein and Rostoker (1955), contains an incorrect additional zero-point energy. A discussion of this problem has been given by Butcher and Ogg (1965).

The nature of the distribution-function calculation is determined by the initial choice of $E(T)$ as opposed to the quantity $E_c(T)$ of classical noise theory and by the form of the density operator for the field. The fact that the c-number algebra involved is identical with that in the solution of a certain classical problem is immaterial; the calculation is physically quantum-mechanical, as has been pointed out by Klauder and Sudarshan (1968, page 192). We are therefore justified in labelling $E(T)$ and our calculation of its fluctuation distribution as quantum-mechanical.

The serious question here is whether in the general case the calculation can be performed, not classically, but semi-classically in the precise sense normally used (Schiff 1955, chap. 10); that is where everything is treated quantum-mechanically but the radiation field, which is treated classically. The answer to this question is now established to be a definite